Math 010 Exam 3
Spring 2026

For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. (6 pts) Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathrm{R} \right\}$ with usual vector addition, but scalar multiplication defined by $c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ y \end{bmatrix}$.

Though several vector space axioms hold, this is not a vector space. Show that one of the following axioms fails.

7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$

Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$.

#8 LHS: $(k+m)\vec{u} = \begin{bmatrix} (k+m)x \\ y \end{bmatrix} = \begin{bmatrix} kx + mx \\ y \end{bmatrix}$

RHS: $k\vec{u} + m\vec{u} = k\begin{bmatrix} x \\ y \end{bmatrix} + m\begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} kx \\ y \end{bmatrix} + \begin{bmatrix} mx \\ y \end{bmatrix} = \begin{bmatrix} kx + mx \\ 2y \end{bmatrix}$

Since $\begin{bmatrix} kx+mx \\ y \end{bmatrix} \neq \begin{bmatrix} kx+mx \\ 2y \end{bmatrix}$ for all y,

that is, LHS $\neq$ RHS, axiom 8 fails.

2. (6 pts) Determine whether

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathrm{R}^3 : x - 2y + z = 0 \right\}$$

is a subspace of R^3.

Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in W$.

$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

$(u_1 + v_1) - 2(u_2 + v_2) + (u_3 + v_3)$

$= u_1 - 2u_2 + u_3 + v_1 - 2v_2 + v_3$

$= 0 + 0$ since $\vec{u}, \vec{v} \in W$.

$= 0 \Rightarrow \vec{u} + \vec{v} \in W$.

Let k be a scalar.

$k\vec{u} = (ku_1, ku_2, ku_3)$

$ku_1 - 2ku_2 + ku_3 = k(u_1 - 2u_2 + u_3)$

$= k(0) = 0$

So $k\vec{u} \in W$.

W is a subspace of $\mathbb{R}^3$.

3. (6 pts) Determine whether

$$\mathbf{w} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

belongs to

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

If it does, express **w** as a linear combination.

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

thus $2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$.

$$\vec{w} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

4. a. (4 pts) Show that the given set is a basis for R^2.

$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

$$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \to \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} \to \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Since this is I, the set is a basis for $\mathbb{R}^2$.

OR $\begin{vmatrix}1 & 1\\1 & -1\end{vmatrix} = -1-1 = -2 \neq 0.$

b. (2 pts) Answer without performing any computations: Is it possible that the given set is a basis? Briefly explain why or why not.

$\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}5\\-1\end{bmatrix}\right\}$

No. A basis for $\mathbb{R}^2$ contains two vectors. Since this set has three, it is linearly dependent.

5. (6 pts) Determine whether each statement is true or false. Justify briefly.

a. The span of two vectors in R^3 is always a plane.

False. If the two vectors are parallel, the span is a line.

b. Every basis for a finite-dimensional vector space has the same number of vectors.

True. Dimension is defined as the number of vectors in a basis.

c. Every linearly independent subset of a vector space V is a basis for V.

False. If it contains fewer vectors than the dimension of V, it will not span V.

6. (8 pts) Find a basis for the subspace of R^3 spanned by

$$\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}.$$

State the dimension of the subspace.

$$\begin{bmatrix}1 & 3 & 1\\2 & 4 & 1\\1 & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 3 & 1\\0 & 2 & 1\\0 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 3 & 1\\0 & 2 & 1\\0 & 0 & 0\end{bmatrix}$$

Since columns 1 & 2 contain pivots, the basis is $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}\right\}$ with

dimension 2.

7. (10 pts) Let $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and let S denote the standard basis for R^2.

a. Find $P_{B \to S}$.

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] \text{ shows } P_{B \to S} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

b. Find $[\mathbf{v}]_B$ for $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & -1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$$[\vec{v}]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

c. Use the matrix you found in part (a) to compute $[\mathbf{v}]_S$. Briefly comment on this result.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = [\vec{v}]_S$$

This is $\vec{v}$ because S is the standard basis for R^2. It verifies that $P_{B \to S}$ is correct.

8. (8 pts) Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$

Find:

a. a basis for $\text{null}(A)$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x_3$, $x_2 = -x_3$. Let $x_3 = t$.

$\vec{x} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix}$, so the basis is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

b. $\text{rank}(A)$

2 (the number of leading 1s)

c. $\text{nullity}(A)$

1 (the dimension of the null space)

9. (8 pts) Prove **one** of the following.

a. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r$ in exactly one way.

b. If a set contains the zero vector, then the set is linearly dependent.

a. Let $\vec{v} \in V$. Then since S spans V, there are c_i such that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$. Suppose there are k_i such that $\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n$. Then

$$\vec{v} - \vec{v} = \vec{0} = (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) - (k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n)$$

$$\Rightarrow (c_1 - k_1)\vec{v}_1 + (c_2 - k_2)\vec{v}_2 + \cdots + (c_n - k_n)\vec{v}_n = \vec{0}$$

Since S is linearly independent, $c_i = k_i$ for all i. Thus the representation is unique.

b. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{0}\}$ and suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n + \vec{0} = \vec{0}$.

Then $\vec{0} = -c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_n\vec{v}_n$, a linear combination of the non zero vectors. The set is dependent.

OR Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{0}\}$.

Then $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n + 1\vec{0} = \vec{0}$

if $c_i = 0$ for $0 \le i \le 1$. Since the linear combination has a nonzero coefficient, the set is dependent.